Indian Statistical Institute
Mid-Semestral Examination 2003-2004
B.Math I Year II Semester
Mathematical Analysis II

Time: 3 hrs | Date:08-03-04 | Max. Marks : 35

Answer all the questions

1. a) Let (X, d) be a metric space. $y_1, y_2, y_3, \ldots$ is a sequence in X converging to y_0. Let $K = \{y_0, y_1, y_2, \ldots\}$. Show that K is a compact set. [3]

   b) Let $f : (X, d) \to (Y, m)$ be any function between metric spaces. If the restriction of f to *each* compact set is continuous show that f is continuous. [2]

2. Let $f : (0, 1) \to R$ be given by $f(x) = \sin \frac{1}{x}$. Show that f is not uniformly continuous. [Hint: If $x_n = \frac{1}{2n\pi}$ and $y_n = \frac{1}{2n\pi + \frac{\pi}{2}}$, what is $f(x_n), f(y_n)$]? [2]

3. a) Define $g : [0, 1] \to R$ by $g(x) = x \sin \frac{1}{x}$ for $x \neq 0$, $g(0) = 0$. Show that g is continuous on $[0, 1]$ [3]

   b) Is g uniformly continuous? give reason. [1]

4. a) Let $f : (X, d) \to (Y, m)$ be uniformly continuous. Let $Z \subset X$. Show that the restriction of f to Z is uniformly continuous [1]

   b) Let $f : R \to R$ be any continuous function f need not be uniformly continuous. For any bounded subset B of R, show that the restriction of f to B is uniformly continuous. [3]

5. a) Let (X, d) be a metric space. $A_1, A_2, A_3, \ldots A_k$ are connected subsets of X such that for each $i = 1, 2, \ldots, k-1$, one has $A_i \cap A_{i+1} \neq$ empty show that $A_1 \cup A_2 \cup \ldots \cup A_k$ is connected [3]

6. Let $A, B, C4$ be subsets of R^2 given by

$$\begin{aligned} A &= \{(x, y) \in R^2 : (x-2)^2 + y^2 = 4\} \\ B &= \{(x, y) \in R^2 : (x+1)^2 + y^2 = 1\} \\ C &= \{(x, y) : y = 0, \quad -1 \leq x \leq 0\} \end{aligned}$$

   Show that $A \cup B \cup C$ is connected. [2]

7. a) Let (X, d) be a metric space. Let S be a connected subset of X. Let $S \subset T$. Assume that for each t in T, there exists a sequence $s_1, s_2, \ldots, s_n, \ldots$ in S such that $s_n \to t$. Show that T is connected [3]

b) Let $A, B \subset R^2$ be given by

$$\begin{aligned} A &= \{(x, y) : \ x > 0, \ y > 0\} \\ B &= \{(k, 0) : \ k = 1, 2, 3, \ldots\} \end{aligned}$$

Show that A and $A \cup B$ are connected [4]

8. Let $f : R^2 \to R$ be given by $f(x_1, x_2) = x_1^2 + \sin x_2$.

a) Calculate $\frac{\partial f}{\partial x_1}$, $\frac{\partial f}{\partial x_2}$. Show that both are continuous functions; $R^2 \to R$ [4]

b) Show that f has total derivative $f'(\underset{\sim}{x})$ for each $\underset{\sim}{x}$ and find it. [2]

c) Let $\underset{\sim}{u}= (1, 1)$. show that $g \ \underset{\sim}{x}= f'(\underset{\sim}{x}, \underset{\sim}{y})$, the directional derivative of f along the direction $\underset{\sim}{u}$ at $\underset{\sim}{x}$, exists and $g : R^2 \to R$ is a continuous function. [2]